MATHEMATICAL LOGIC HOMEWORK 6

Due Monday April 15.

Let $\phi_{prov-T}(e)$ be the formula in the language of arithmetic which states that e codes a formula ϕ_e and $T \vdash \phi_e$. Also for a formula ϕ, $\ulcorner \phi \urcorner$ denotes the Gödel number of ϕ

Problem 1. *Determine which of the following is true of false for an arbitrary sentence τ, and prove your answer.*

(1) *If $PA \vdash \tau$, then $PA \vdash \phi_{prov-PA}(\ulcorner \tau \urcorner)$.*
(2) *$PA \vdash \tau \to \phi_{prov-PA}(\ulcorner \tau \urcorner)$.*
(3) *$PA \vdash \phi_{prov-PA}(\ulcorner \tau \urcorner) \to \tau$.*

Problem 2. *Show that if ϕ is Π_1, then $PA + Con_{PA}(\ulcorner \phi \urcorner) \vdash \phi$. Here $Con_{PA}(\ulcorner \phi \urcorner)$ is the formula stating that PA does not refute ϕ i.e. $\neg \phi_{prov-PA}(\ulcorner \neg \phi \urcorner)$.*

Problem 3. *Suppose $f : \mathbb{N} \to \mathbb{N}$ is total recursive. Prove that $A = \bigcup_n W_{f(n)}$ is r.e.*

Problem 4. *(a) (Reduction) Suppose that A, B are r.e. sets. Prove that there are disjoint r.e. sets A_0, B_0, such that $A_0 \subset A$, $B_0 \subset B$, and $A_0 \cup B_0 = A \cup B$.*

(b) (Separation) Suppose A and B are disjoint Π^0_1 sets. Prove that there is a recursive set C, such that $A \subset C$ and $C \setminus B = \emptyset$.

Problem 5. *Prove that $\{e \mid W_e \neq \emptyset\}$ is Σ^0_1 complete.*